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# A classification of 2D random Dirac fermions

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**Abstract**
We present a detailed classification of random Dirac Hamiltonians in two spatial dimensions based on the implementation of discrete symmetries. Our classification is slightly finer than that of random matrices and contains 13 classes. We also extend this classification to non-Hermitian Hamiltonians with and without a Dirac structure.

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## 1. Introduction and results

Recently, there has been an increasing interest in two-dimensional localization problems whose behaviour differs from generic Anderson localization [1]. To mention but a few examples, there are investigations of the quantum Hall transition [2–5], of quasi-particle localization in systems with degenerate Fermi surfaces [6] and dirty superconductors [7–10], and studies of hopping models on bipartite lattices [11, 12].

Universality classes of localization/delocalization transitions depend largely on their discrete symmetries. Thus the Wigner–Dyson classification of random Hermitian matrices [13, 14] plays a significant role. Localization in superconductors led Altland and Zirnbauer to significantly extend the Wigner–Dyson classification by incorporating particle–hole and chirality symmetries of the matrices [15].

Most of the localization problems mentioned above may be formulated as spectral problems for Dirac-like Hamiltonians in two spatial dimensions and many of them differ from generic Anderson localization in that they exhibit a singular density of states at the critical point.

In this paper, we present a somewhat detailed classification of such Dirac Hamiltonians in two dimensions. This classification potentially differs from that of random matrices because in the latter no structure is imposed on the matrices—and matrices differing by unitary similarity transformations are treated as equivalent—whereas in the former the Hamiltonians *a priori*

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possess a Dirac form. This may have two opposite effects: either some random matrix classes may not be realized by Dirac operators, or Dirac Hamiltonians belonging to the same random matrix class may not be equivalent if the unitary similarity transformation relating them does not preserve the imposed Dirac structure. Although some models of random Dirac fermions have already been identified with Altland–Zirnbauer classes, the correspondences have not been fully established and one motivation of our classification is a more complete dictionary.

The essential difference between these two classifications is a doubling of the chiral classes. Namely, each of the three chiral classes of random matrices is split into two distinct Dirac Hamiltonian classes. Thus, there exist random Dirac Hamiltonians which belong to the identical matrix class but which may exhibit different localization phenomena. This makes clear that the physics of the two-dimensional random Dirac fermions is richer than expected from the random matrix theory. These differences do not occur for Schrödinger-like Hamiltonians of the form $H = -\nabla_x^2 + V(x)$ with $V(x)$ being local random matrices.

Since each of the ten classes of random matrices has been associated with a $\sigma$-model on a symmetric space, an interesting question concerns how the $\sigma$-models can incorporate the finer classification described in this paper. One possible answer is that $\sigma$-models in two dimensions—towards which the effective field theory describing these transitions may flow—support a topological $\theta$-term or Wess–Zumino term. These terms may render the theories critical or conformally invariant. This scenario has recently been checked in [33] in a large $N$ approach using controlled approximations. The two chiral classes of Dirac Hamiltonians are distinguished by the presence or absence of a WZW term, so that in each pair of chiral classes one is conformally invariant. This illustrates the relevance of our finer classification.

We consider Dirac Hamiltonians $H = (\tau_x p_x + \tau_y p_y)/2 + \vec{\tau}\cdot\vec{W} + W_0$, where $\vec{\tau}$ are Pauli matrices[5], $p_{x,y} = -\mathrm{i}\partial_{x,y}$, and $W_0$ and $\vec{W}$ are generalized masses or potentials, and are matrices acting on an isospin sector. Introducing complex coordinates, $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, and $\partial_z = (\partial_x - \mathrm{i}\partial_y)/2$, $\partial_{\bar{z}} = (\partial_x + \mathrm{i}\partial_y)/2$, after a unitary transformation one generally obtains the following $2 \times 2$ block structure:

$$H = \begin{pmatrix} V_+ + V_- & -\mathrm{i}\partial_{\bar{z}} + A_{\bar{z}} \\ -\mathrm{i}\partial_z + A_z & V_+ - V_- \end{pmatrix}. \tag{1.1}$$

Here $A_z$, $A_{\bar{z}}$ and $V_\pm$ are random matrices depending on the spatial coordinates $x, y$ and belonging to some statistical ensemble.

As usual, the classes are sets of Hamiltonians with specific transformation properties under some discrete symmetries. For Dirac Hamiltonians (1.1), the simplest symmetries are chiral, particle–hole and time-reversal symmetry, which relate the Hamiltonian $H$ to $-H$, its transpose $H^T$ and its complex conjugate $H^*$, respectively. We demand that these transformations are implemented by unitary transformations and that their actions on the Hamiltonian square to one. They also should preserve the form (1.1) of the Dirac Hamiltonian. Hence we consider the following transformations:

$$P \text{ symmetry:} \qquad H = -PHP^{-1} \qquad P = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \qquad PP^\dagger = 1 \qquad P^2 = 1 \tag{1.2}$$

$$C \text{ symmetry:} \qquad H = \epsilon_c CH^TC^{-1} \qquad C = \begin{pmatrix} 0 & \sigma \\ -\epsilon_c\sigma & 0 \end{pmatrix} \qquad CC^\dagger = 1 \qquad C^T = \pm C \tag{1.3}$$

$$K \text{ symmetry:} \qquad H = \epsilon_k KH^*K^{-1} \qquad K = \begin{pmatrix} 0 & \kappa \\ -\epsilon_k\kappa & 0 \end{pmatrix} \qquad KK^\dagger = 1 \qquad K^T = \pm K \tag{1.4}$$

[5] Pauli matrices will be denoted by $\vec{\tau}$ or $\vec{\sigma}$ depending on which space they are acting. Our convention is $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$.

**Table 1.** Random Dirac Hamiltonian classes.

| | Random matrix class | Time-reversal invariant | Particle–hole symmetry | Chirality | Symmetry group |
|---|---|---|---|---|---|
| Class **0** | $\mathbf{A}$ = GUE | no | no | no | $U(n)$ |
| Class **1** | $\mathbf{A}_{\mathrm{III}}$ = chiral GUE | no | no | yes | $U(n)$ |
| Class **2** | $\mathbf{A}_{\mathrm{III}}$ = chiral GUE | no | no | yes | $U(n) \times U(n)$ |
| Class $\mathbf{3}_+$ | $\mathbf{A}_{\mathrm{II}}$ = GSE | yes | no | no | $O(n)$ |
| Class $\mathbf{3}_-$ | $\mathbf{D}$ | no | yes | no | $O(n)$ |
| Class $\mathbf{4}_+$ | $\mathbf{A}_{\mathrm{I}}$ = GOE | yes | no | no | $Sp(2n)$ |
| Class $\mathbf{4}_-$ | $\mathbf{C}$ | no | yes | no | $Sp(2n)$ |
| Class **5** | $\mathbf{D}_{\mathrm{III}}$ = chiral GOE | yes | yes | yes | $O(n)$ |
| Class **6** | $\mathbf{C}_{\mathrm{I}}$ = chiral GSE | yes | yes | yes | $Sp(2n)$ |
| Class **7** | $\mathbf{D}_{\mathrm{III}}$ = chiral GOE | yes | yes | yes | $O(n) \times O(n)$ |
| Class **8** | $\mathbf{C}_{\mathrm{I}}$ = chiral GSE | yes | yes | yes | $Sp(2n) \times Sp(2n)$ |
| Class $\mathbf{9}_+$ | $\mathbf{D}_{\mathrm{I}}$ | yes | yes | yes | $U(n)$ |
| Class $\mathbf{9}_-$ | $\mathbf{C}_{\mathrm{II}}$ | yes | yes | yes | $U(n)$ |

where $\epsilon_c = \pm 1$. Type $P$ symmetries are commonly referred to as chirality symmetries, $C$ expresses a particle–hole symmetry and $K$ time-reversal symmetry. For Hermitian Hamiltonians, since $H^T = H^*$, $C$ and $K$ symmetries are identical and we will only talk about $C$ symmetry, where $\epsilon_c = +1$ will be interpreted as time-reversal symmetry and $\epsilon_c = -1$ will be referred to as particle–hole symmetry.

We found 13 distinct classes of Hermitian Dirac Hamiltonians, listed in equations (2.16)–(2.25). This classification, which is presented in section 2, is slightly finer than that of random Hermitian matrices [15]. See table 1 for comparison. (The numbering of the classes has no special meaning.)

Let us mention a few well-known realizations of the classes of table 1. Classes **0** (GUE), $\mathbf{3}_+$ (GSE) and $\mathbf{4}_+$ (GOE) are the usual Wigner–Dyson classes. The $U(1)$ model of class **0** was introduced in [5] in connection with the quantum Hall transition. The $U(n)$ case of class **0** appeared in [6] for describing systems with degenerate Fermi points. The chiral classes **1** (chGUE), **5** (chGOE) and **6** (chGSE) are realized by Dirac operators coupled only to random gauge potentials [19]. The $U(n)$ model of class **1** was applied to dirty d-wave superconductors in [7]. The pure $U(1)$ random gauge potential is still not completely solved due to some recently recognized non-perturbative effects [16–18]; the non-Abelian cases have been solved by various methods [7, 20, 21]. The chiral class **2** (chGUE) is realized by the Gade–Wegner hopping models [11]. (See [22] for a recent numerical study.) The class $\mathbf{3}_-$ (D) [23] and class $\mathbf{4}_-$ (C) [24] appeared in the context of dirty superconductors with broken time-reversal symmetry. Numerical analysis of the *sp*(2) model of class $\mathbf{4}_-$ (C), the so-called spin quantum Hall effect, was performed in [25] and exact results were obtained by mapping it to percolation [26].

Non-Hermitian random Hamiltonians have recently been used in the description of various phenomena (see e.g. [27]). In section 3 we extend our classification to non-Hermitian Hamiltonians. In addition to type $P$, $C$ and $K$ symmetries, we may consider a $Q$ symmetry relating $H$ to its adjoint:

$$Q \text{ symmetry}: \qquad H = \epsilon_q Q H^\dagger Q^{-1} \quad Q = \begin{pmatrix} \xi & 0 \\ 0 & \epsilon_q \xi \end{pmatrix} \quad Q Q^\dagger = 1 \quad Q^2 = 1. \tag{1.5}$$

Imposition of these symmetries selects reality conditions on the potentials $A_z$, $A_{\bar{z}}$ and $V_\pm$. Of course, the type $Q$ symmetry with $\epsilon_q = 1$ and $\xi = 1$ simply means that $H$ is Hermitian. From any non-Hermitian Dirac operator one may naturally define a Hermitian operator, denoted by

$\mathcal{H}$, by doubling the Hilbert space on which it acts (see equations (3.4)). The latter Hamiltonian is then a representative of the chiral class we have indexed as class **2** (chGUE). However, as we explain, the classification of the non-Hermitian Dirac operators $H$ is finer, and more involved, than that of their doubled companions $\mathcal{H}$. We find a total of 87 universality classes.

As a byproduct of our analysis, we are easily able to classify non-Hermitian random matrices without a Dirac structure. This leads to 43 classes.

## 2. Classification of Hermitian Dirac Hamiltonians

We first consider the Hermitian Hamiltonians which require $V_\pm^\dagger = V_\pm$ and $A_z^\dagger = A_{\bar{z}}$. Let us define a 'minimal class' as a class of Dirac Hamiltonians which cannot be simultaneously block diagonalized. For such Hamiltonians there exists no fixed unitary matrix $S$ that commutes with the Hamiltonian, $H = SHS^{-1}$, and preserves the Dirac structure, which requires $S = \mathrm{diag}(s, s)$. The existence of an integral of motion, such as spin, implies such an $S$ and the resulting Hamiltonian is thus not minimal according to our definition. Rather, our ensembles apply to each block with fixed quantum numbers of the integrals of motion.

### 2.1. Compatible symmetries

Only type $P$ and type $C$ symmetries are relevant for Hermitian Hamiltonians. We first need to classify the compatible operators $P$ and $C$, or preferably the compatible $\gamma$ and $\sigma$. It is important to bear in mind that what is meaningful is the group generated by these symmetries. For instance, if the Hamiltonian possesses both $P$ and $C$ symmetries, then it automatically has another $C$-type symmetry $C'$:

$$H = \epsilon_c' C' H^T C'^{-1} \qquad C' = PC \qquad \epsilon_c' = -\epsilon_c. \tag{2.1}$$

For Hermitian Hamiltonians, since $C'$ can be interpreted as a time-reversal (particle–hole) symmetry if $\epsilon_c = -1$ ($\epsilon_c = +1$), the classes with both $P$ and $C$ symmetries thus automatically have chirality, particle–hole and time-reversal symmetry.

The operators $P$ and $C$ are defined up to dilatations by scalars and up to unitary changes of the basis $H \to UHU^\dagger$, which preserve the form of the Dirac Hamiltonians. This requires $U = \mathrm{diag}(u, u)$. On $\gamma$ and $\sigma$, this translates into

$$\gamma \to u\gamma u^\dagger \qquad \sigma \to u\sigma u^T \tag{2.2}$$

with $u$ being unitary. The unitarity and the order two constraints on $P$ and $C$ imply

$$\gamma\gamma^\dagger = 1 \qquad \gamma^2 = 1 \qquad \sigma\sigma^\dagger = 1 \qquad \sigma^T = \pm\sigma. \tag{2.3}$$

These conditions are covariant under the gauge transformations (2.2).

First let us only impose a type $P$ symmetry. Modulo (2.2) we can reduce $\gamma$ to a diagonal matrix with only $\pm 1$ on the diagonal. We may thus choose

$$\text{case (1)}: \qquad \gamma = 1 \tag{2.4}$$

$$\text{case (2)}: \qquad \gamma = \sigma_z \otimes 1. \tag{2.5}$$

In the second case, we assumed for simplicity that the numbers +1 and $-1$ in $\gamma$ are equal, but this could be generalized.

Let us now impose only a type $C$ symmetry. Up to the transformations (2.2), there are two (standard) cases [13] depending on the condition $\sigma^T = \pm\sigma$:

$$\text{case (3)}: \qquad \sigma = 1 \tag{2.6}$$

$$\text{case (4)}: \qquad \sigma = \mathrm{i}\sigma_y \otimes 1. \tag{2.7}$$

Indeed, assume that $\sigma^T = \sigma$. Then, since $\sigma$ is also unitary, $\sigma\sigma^* = 1$ and its real and imaginary parts commute and are both symmetric. They can be simultaneously diagonalized by a real orthogonal matrix $o$, so that $\sigma = o\delta o^T$ with $\delta$ a diagonal unitary matrix. Hence, $\sigma = uu^T$, with $u = o\delta^{1/2}$ unitary, and $\sigma$ is equivalent to the identity modulo (2.2). The argument is similar for $\sigma^T = -\sigma$.

Next we impose simultaneously type $P$ and type $C$ symmetries. These symmetries have to be compatible in the sense that their actions on Hamiltonians should commute. For generic Hamiltonians this requires that $C \propto PCP^T$, or $\sigma \propto \gamma\sigma\gamma^T$. This condition is covariant under transformations (2.2). So we may choose a basis in which $\gamma$ is diagonal and we restrict ourselves to the two cases (2.4), (2.5). We then have two sub-cases corresponding to the two possible values $\pm 1$ of the proportionality coefficient in the above equation, so that $\sigma$ either commutes or anticommutes with $\gamma$:

$$\sigma = \pm\gamma\sigma\gamma^T \quad \Rightarrow \quad [\sigma, \gamma] = 0 \quad \text{or} \quad \{\sigma, \gamma\} = 0. \tag{2.8}$$

If $\gamma = 1$, $\sigma$ automatically commutes with it and we obtain

$$\text{case (5)}: \qquad \gamma = 1 \qquad \sigma = 1 \tag{2.9}$$

$$\text{case (6)}: \qquad \gamma = 1 \qquad \sigma = \mathrm{i}\sigma_y \otimes 1. \tag{2.10}$$

If $\gamma = \sigma_z \otimes 1$, we have to separately consider the two possibilities in (2.8). The transformations (2.2) have to preserve the form of $\gamma$ so that $u$ has to be block diagonal, $u = \mathrm{diag}(u_1, u_2)$ with $u_{1,2}$ unitary. When $[\sigma, \gamma] = 0$, $\sigma$ also has to be block diagonal, $\sigma = \mathrm{diag}(\sigma_1, \sigma_2)$. As above, modulo (2.2) with $u = \mathrm{diag}(u_1, u_2)$, it can be reduced to $\sigma = 1$ if $\sigma^T = \sigma$ and $\sigma = \mathrm{i}\sigma_y \otimes 1$ if $\sigma^T = -\sigma$. Thus, we get two possibilities,

$$\text{case (7)}: \qquad \gamma = \sigma_z \otimes 1 \qquad \sigma = 1_2 \otimes 1 \tag{2.11}$$

$$\text{case (8)}: \qquad \gamma = \sigma_z \otimes 1_2 \otimes 1 \qquad \sigma = 1_2 \otimes \mathrm{i}\sigma_y \otimes 1. \tag{2.12}$$

When $\{\sigma, q\} = 0$, $\sigma$ has to be block off-diagonal, so that $\sigma = \left(\begin{smallmatrix} 0 & s \\ \pm s^T & 0 \end{smallmatrix}\right)$, with $s$ being unitary, depending on whether $\sigma$ is symmetric or antisymmetric. The gauge transformations (2.2) then become $s \to u_1\, s\, u_2^T$ with $u_{1,2}$ being unitary, and any unitary $s$ is gauge equivalent to the identity. This gives two cases:

$$\begin{aligned} &\text{case (9)}: \qquad \gamma = \sigma_z \otimes 1 \qquad \sigma = \mathrm{i}\sigma_y \otimes 1 \\ &\text{case}(9'): \qquad \gamma = \sigma_z \otimes 1 \qquad \sigma = \sigma_x \otimes 1. \end{aligned} \tag{2.13}$$

Cases (9) and (9′) turn out to be equivalent because the type $C$ symmetry of one of the two cases follows from the product of type $P$ and type $C$ symmetries of the other case.

Finally, let us consider more combinations of type $P$ or $C$ symmetries. If we impose two symmetries of type $P$, their product commutes with the Hamiltonians and this system thus does not correspond to a minimal class. Next, consider imposing two compatible symmetries of type $C$ with signs $\epsilon_{c1}$ and $\epsilon_{c2}$. If the product $\epsilon_{c1}\epsilon_{c2} = -1$, their product (see equation (2.1)) makes a type $P$ symmetry. Thus two type $C$ symmetries with opposite $\epsilon_c$ signs are equivalent to type $P$ and type $C$ symmetries which we have already classified. If the $\epsilon_c$ signs are equal, the product of the two type $C$ symmetries commutes with the Hamiltonians and this system is not minimal. More generally, considering more combinations of type $P$ and type $C$ symmetries does not lead to new minimal classes.

*2.2. List of classes*

In this subsection we present the detailed structure of the resulting classes of Hamiltonians. The type $P$ and $C$ symmetries impose the following relations on the generalized potentials and masses:

$$P \text{ symmetry}: \qquad \gamma A_z = A_z\gamma \qquad \gamma A_{\bar z} = A_{\bar z}\gamma \qquad \gamma V_\pm + V_\pm\gamma = 0 \tag{2.14}$$

$$C \text{ symmetry}: \qquad \sigma A_z^T + A_z\sigma = 0 \qquad \sigma A_{\bar z}^T + A_{\bar z}\sigma = 0 \qquad \sigma V_\pm^T = \pm\epsilon_c V_\pm\sigma. \tag{2.15}$$

They possess a simple interpretation as they indicate that $A_z$ and $A_{\bar z}$ belong to an orthogonal or symplectic Lie algebra depending on whether $\sigma$ is symmetric or antisymmetric. The compatibility relations (2.8), $\sigma = \pm\gamma\sigma\gamma^T$, ensure that constraints (2.14) and (2.15) may be imposed simultaneously. When imposing both type $P$ and type $C$ symmetries, one generates all relations obtained by successive applications of these symmetries, i.e. all relations associated with elements of the group generated by type $P$ and $C$ symmetries are imposed. As a consequence, there could be different presentations of the same class depending on which generators of this group one selects. For example, given a type $P$ and a type $C$ symmetry with a sign $\epsilon_c$, their product is again a type $C$ symmetry but with an opposite sign $-\epsilon_c$ (see equation (2.1)).

Solutions of the constraints (2.14) and (2.15) for the set of compatible $\gamma$ and $\sigma$ give the following minimal classes:

$$\text{class } \mathbf{0}: \qquad A_{\bar z}, A_z \in gl(n) \qquad V_\pm \in gl(n). \tag{2.16}$$

$$\text{class } \mathbf{1}: \qquad A_z \in gl(n) \qquad V_\pm = 0. \tag{2.17}$$

$$\text{class } \mathbf{2}: \qquad A_z = \text{diag}(a_+, a_-) \qquad a_\pm \in gl(n) \qquad V_\pm = \begin{pmatrix} 0 & v_\pm \\ w_\pm & 0 \end{pmatrix} \qquad v_\pm, w_\pm \in gl(n). \tag{2.18}$$

$$\text{class } \mathbf{3}_{\epsilon_c}: \qquad A_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -A_z^T \in so(n) \qquad a = -a^T \qquad b = -c^T \qquad d = -d^T \qquad V_{-\epsilon_c} = -V_{-\epsilon_c}^T \in so(n) \qquad V_{\epsilon_c} = V_{\epsilon_c}^T \in gl(n) \setminus so(n). \tag{2.19}$$

$$\text{class } \mathbf{4}_{\epsilon_c}: \qquad A_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\sigma_y A_z^T\sigma_y \in sp(2n) \qquad a = -d^T \qquad b = b^T \qquad c = c^T \qquad V_{-\epsilon_c} = -\sigma_y V_{-\epsilon_c}^T\sigma_y \in sp(2n) \qquad V_{\epsilon_c} = \sigma_y V_{\epsilon_c}^T\sigma_y \in gl(2n) \setminus sp(2n). \tag{2.20}$$

$$\text{class } \mathbf{5}: \qquad A_z = -A_z^T \in so(n) \qquad V_\pm = 0. \tag{2.21}$$

$$\text{class } \mathbf{6}: \qquad A_z = -\sigma_y A_z^T\sigma_y \in sp(2n) \qquad V_\pm = 0. \tag{2.22}$$

$$\text{class } \mathbf{7}: \qquad A_z = \text{diag}(a_+, a_-) \qquad a_\pm = -a_\pm^T \in so(n) \qquad V_\pm = \begin{pmatrix} 0 & v_\pm \\ w_\pm & 0 \end{pmatrix} \qquad v_{\pm\epsilon_c} = \pm w_{\pm\epsilon_c}^T. \tag{2.23}$$

$$\text{class } \mathbf{8}: \qquad A_z = \text{diag}(a_+, a_-) \qquad a_\pm = -\sigma_y a_\pm^T\sigma_y \in sp(2n) \qquad V_\pm = \begin{pmatrix} 0 & v_\pm \\ w_\pm & 0 \end{pmatrix} \qquad v_{\pm\epsilon_c} = \pm\sigma_y w_{\pm\epsilon_c}^T\sigma_y. \tag{2.24}$$

$$\text{class } \mathbf{9}_{\epsilon_c}: \qquad A_z = \text{diag}(a, -a^T) \qquad a \in gl(n) \qquad V_{\pm\epsilon_c} = \begin{pmatrix} 0 & v_{\pm\epsilon_c} \\ w_{\pm\epsilon_c} & 0 \end{pmatrix} \qquad v_{\pm\epsilon_c} = \mp v_{\pm\epsilon_c}^T \qquad w_{\pm\epsilon_c} = \mp w_{\pm\epsilon_c}^T. \tag{2.25}$$

(The labels 1–9 refer to the cases 1–9 listed in the previous subsection.) The Hermiticity constraints $A_{\bar z} = A_z^\dagger$, $V_\pm = V_\pm^\dagger$, $v_\pm^\dagger = w_\pm$, are implicit in this list. (The Hermiticity constraint is not made explicit for the purpose of describing the non-Hermitian classes in the next section.) The index $\epsilon_c$ refers to one of the two possible values $\epsilon_c = \pm$, and the absence of such an index means that this value is irrelevant. The first class corresponds to generic Dirac Hamiltonians with no constraints imposed. There are a few degeneracies when solving equations (2.14) and (2.15). As expected, cases 9 and 9′, equation (2.13), yield the same solutions; also realizations of the cases 5 and 6, equations (2.9) and (2.10) are independent of the choice of the sign $\epsilon_c$. Realizations of the cases $7_\pm$ are also equivalent because they correspond to different presentations of the same class. Indeed, consider equation (2.11) in case 7. The product of its type $P$ and type $C$ defining symmetries produces a new type $C$ symmetry with opposite sign and with $\sigma' = \sigma_z \otimes 1$. It is gauge equivalent to the original symmetries as $\sigma' \simeq u_7 \sigma' u_7^T = 1_2 \otimes 1$ with $u_7 = \mathrm{diag}(1, \mathrm{i}) \otimes 1$. Hence cases $7_+$ and $7_-$ are gauge equivalent and we give the two presentations in the above list. The corresponding realizations are related by similarity transformations $H \to H' = UHU^\dagger$ with $U = \mathrm{diag}(u_7, u_7)$, so that $a'_\pm = \pm a_\pm$ and $v'_\pm = -\mathrm{i}v_\pm$, $w'_\pm = \mathrm{i}w_\pm$. A similar argument applies to the cases $8_\pm$ showing again that they are equivalent presentations of the same class.

One of the origins of the distinction between the above classification and the classification of random matrices arises from a difference in the notion of equivalent classes of compatible symmetries. For random matrices, some of the cases with $\gamma = 1$ or $\gamma = \sigma_z \otimes 1$ are considered as equivalent because they correspond to the same operator $P$ up to re-shuffling of the lines and columns, while in the present classification they yield different classes because we impose the $2 \times 2$ block structure (1.1) on the Hamiltonians[6]. Thus the classification of the random Dirac operators is a bit finer than that of random matrices, as summarized in table 1.

### 2.3. Symmetry groups, disorder measures and super-symmetric effective actions

Each class is stable under the action of a symmetry group, whose elements act on the Hamiltonians by conjugation such that their form imposed by equations (2.16)–(2.25) is preserved. Elements $G$ of the symmetry groups satisfy[7]

$$G\gamma G^{-1} = \gamma \qquad G\sigma G^T = \sigma. \tag{2.26}$$

The list of these groups is given in table 1. For classes **2**, **7**, **8**, in which the symmetry group is a product of two subgroups, the embedding is diagonal with $G = \mathrm{diag}(g_+, g_-)$ where $g_\pm$ belong either to $U(n)$, $O(n)$ or $Sp(2n)$. In class **9** the embedding is $G = \mathrm{diag}(g, g^T)$ with $g \in U(n)$.

The symmetry group may be used to specify the disorder measures in each class, which we assume to be Gaussian, with zero mean, and local. The measures are then fixed by requiring them to be invariant under the symmetry group. The list of all quadratic invariants for each class is the following:

class **0**: $\mathrm{tr}(A_z A_{\bar z})$, $\mathrm{tr}(A_z)\mathrm{tr}(A_{\bar z})$, $\mathrm{tr}(V_\pm^2)$, $\mathrm{tr}(V_\pm)^2$, $\mathrm{tr}(V_\pm V_\mp)$, $\mathrm{tr}(V_\pm)\mathrm{tr}(V_\mp)$

class **1**: $\mathrm{tr}(A_z A_{\bar z})$, $\mathrm{tr}(A_z)\mathrm{tr}(A_{\bar z})$

class **2**: $\mathrm{tr}(a_\pm \bar a_\pm)$, $\mathrm{tr}(a_\pm)\mathrm{tr}(\bar a_\pm)$, $\mathrm{tr}(v_\pm^\dagger v_\pm)$, $\mathrm{tr}(v_\pm^\dagger v_\mp)$

class $\mathbf{3}_{\epsilon_c}$: $\mathrm{tr}(A_z A_{\bar z})$, $\mathrm{tr}\left(V_{\pm\epsilon_c}^2\right)$, $\mathrm{tr}(V_{\epsilon_c})^2$

[6] The classification of random matrices may be reread from the previous classification by considering that $\gamma$ and $\sigma$ implement directly the discrete type $P$ and $C$ symmetries. Only the cases with $\gamma = \sigma_z \otimes 1$ are then relevant since $\gamma = 1$ is trivial. As it should be, we are left with ten classes **0**, **2**, $\mathbf{3}_\pm$, $\mathbf{4}_\pm$, **7**, **8**, $\mathbf{9}_\pm$. However, in table 1, the random matrix classes refer to those defined by $C$ and $P$ and not by $\gamma$ and $\sigma$ (see the appendix).

[7] One may extend slightly the symmetry group by discrete groups, made of $\mathbf{Z}_2$ factors, by allowing signs in equation (2.26), $G\gamma G^{-1} = \pm\gamma$, $G\sigma G^T = \pm\sigma$.

class $\mathbf{4}_{\epsilon_c}$: $\operatorname{tr}(A_z A_{\bar z})$, $\operatorname{tr}\big(V^2_{\pm\epsilon_c}\big)$, $\operatorname{tr}(V_{\epsilon_c})^2$

class $\mathbf{5}$: $\operatorname{tr}(A_z A_{\bar z})$

class $\mathbf{6}$: $\operatorname{tr}(A_z A_{\bar z})$

class $\mathbf{7}$: $\operatorname{tr}(a_\pm \bar a_\pm)$, $\operatorname{tr}(v^\dagger_\pm v_\pm)$

class $\mathbf{8}$: $\operatorname{tr}(a_\pm \bar a_\pm)$, $\operatorname{tr}(v^\dagger_\pm v_\pm)$

class $\mathbf{9}_{\epsilon_c}$: $\operatorname{tr}(a\bar a)$, $\operatorname{tr}(a)\operatorname{tr}(\bar a)$, $\operatorname{tr}(v^\dagger_\pm v_\pm)$, $\operatorname{tr}(v^\dagger_\pm v_\mp)$.

To preserve rotation invariance we only list the invariants which couple $A_z$ to $A_{\bar z}$ and $V_\pm$ to itself or to $V_\mp$. Couplings between $V_\pm$ and $V_\mp$ break the symmetry under reflection $x \to x$, $y \to -y$.

These ensembles may be analysed using the supersymmetric method [28]. For each class, this leads to an effective field theory description with the number of coupling constants equal to the number of invariants. These coupling constants, which measure the strength of the disorder, parametrize perturbations of the free field theory valid in the absence of disorder. In two dimensions, all of these effective field theories can be formulated as left–right current–current perturbations, where the couplings are marginal [20]. This means that in the effective field theory the coupling constants are dual to operators of scaling dimension two. The discrete symmetry defining the classes plus the global invariance under the symmetry group should ensure that for each class the effective field theory is perturbatively renormalizable, i.e. no additional marginal operators beyond those dual to the coupling constants are generated by the renormalization procedure. This aspect can be studied using the all-orders $\beta$ function proposed in [29], as was done for class $\mathbf{0}$ and for class $4_-$ at $n = 1$ [30].

We can easily describe the global Lie superalgebra symmetry of the effective field theories. The unperturbed conformal field theory has an $osp(2N|2N)$ current algebra symmetry at level 1 where $N$ is the number of fermions, i.e. $N = n$ for classes **0, 1, 3** and **5**, $N = 2n$ for classes **2, 4, 6, 7** and **9**, and $N = 4n$ for class **8**. In the supersymmetric effective theory the global supersymmetry is smaller and corresponds to the Lie superalgebraic extension $\mathcal{G}$ of the symmetry groups listed in table 1. The bosonic group $U(n)$ is extended to $gl(n|n)$, $O(n)$ to $osp(n|n)$ and $sp(2n)$ to $osp(2n|2n)$, so that $\mathcal{G} = gl(n|n)$ for classes **0, 1, 9**, $\mathcal{G} = osp(n|n)$ for classes **3, 5**, $\mathcal{G} = osp(2n|2n)$ for classes **4, 6** and a tensor product of these supergroups for classes **2, 7, 8**.

## 3. Classification of non-Hermitian Dirac Hamiltonians

The classification of non-Hermitian Dirac Hamiltonians we present is based on implementing discrete symmetries of type $P$ or $C$, equations (1.2), (1.3), and of type $Q$ and $K$ defined in equations (1.5), (1.4). Since this parallels closely what we have done for Hermitian Hamiltonians we only sketch the main features of the classification. The order two constraints on type $Q$ and type $K$ symmetry are

$$\xi\xi^\dagger = 1 \qquad \xi^2 = 1 \qquad \kappa\kappa^\dagger = 1 \qquad \kappa^T = \pm\kappa.$$

We consider Hamiltonians up to unitary changes of the basis, $H \to UHU^\dagger$, with $U = \operatorname{diag}(u, u)$, which act on $Q$ and $K$ as

$$\xi \to u\xi u^\dagger \qquad \kappa \to u\kappa u^T. \tag{3.1}$$

As before we define minimal classes as those whose Hamiltonians do not commute with a fixed matrix preserving their Dirac structure.

Imposing type $Q$ or $K$ symmetries amounts to imposing some reality conditions on the Hamiltonians, i.e. some reality properties of $A_z$, $A_{\bar z}$ and $V_\pm$:

$$Q\ \text{symmetry:} \qquad \xi A_{\bar z}^\dagger = A_z \xi \qquad \xi V_\pm^\dagger = \epsilon_q V_\pm \xi \tag{3.2}$$

$$K\ \text{symmetry:} \qquad \kappa A_{\bar z}^* + A_z \kappa = 0 \qquad \kappa V_\pm^* = \pm\epsilon_k V_\pm \kappa. \tag{3.3}$$

Redefining $H \to \mathrm{i}H$ modifies the signs $\epsilon_q$ and $\epsilon_k$ in equations (1.5), (1.4); however, this redefinition ruins the Dirac structure (1.1) and we shall thus not allow it.

The classification of non-Hermitian Hamiltonians may be translated into detailed properties of the Hermitian Hamiltonians $\mathcal{H}$ obtained by doubling the Hilbert spaces on which the Dirac Hamiltonians $H$ are acting:

$$\mathcal{H} = \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}. \tag{3.4}$$

These doubled Hamiltonians are always chiral as they anticommute with $\Gamma_5 = \mathrm{diag}(1, -1)$. Any similarity transformation $H \to UHU^{-1}$ is mapped into $\mathcal{H} \to \mathcal{U}\mathcal{H}\mathcal{U}^\dagger$ with $\mathcal{U} = \mathrm{diag}(U, U^{\dagger -1})$. Demanding that these transformations also act by similarity on $\mathcal{H}$ imposes $U$ to be unitary. When no discrete symmetries are imposed, the doubled Hamiltonians $\mathcal{H}$ are always elements of class **2**, which is embedded in the chiral GUE class. Indeed, up to the re-shuffling of lines and columns, they may be presented as

$$\mathcal{H} \simeq \mathcal{H}_d \equiv \begin{pmatrix} 0 & V_+^\dagger + V_-^\dagger & -\mathrm{i}\partial_{\bar z} + A_z^\dagger & 0 \\ V_+ + V_- & 0 & 0 & -\mathrm{i}\partial_{\bar z} + A_{\bar z} \\ -\mathrm{i}\partial_z + A_z & 0 & 0 & V_+ - V_- \\ 0 & -\mathrm{i}\partial_z + A_{\bar z}^\dagger & V_+^\dagger - V_-^\dagger & 0 \end{pmatrix}. \tag{3.5}$$

The dictionary is thus $a_+ = A_z$, $a_- = A_{\bar z}^\dagger$ and $2v_\pm = (V_+^\dagger \pm V_+) + (V_-^\dagger \mp V_-)$.

On $\mathcal{H}$, both type $P$ and $Q$ symmetries act as chiral transformations, $\mathcal{H} \to -\mathcal{P}\mathcal{H}\mathcal{P}^{-1}$ with $\mathcal{P} = \mathrm{diag}(P, P)$ and $\mathcal{H} \to \epsilon_q \mathcal{Q}\mathcal{H}\mathcal{Q}^{-1}$ with $\mathcal{Q} = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}$. Thus, $\mathcal{H}$ may be block diagonalized if $H$ is $Q$- or $P$-symmetric. Indeed, if $H$ is $Q$-symmetric with $\epsilon_q = +1$ then $\mathcal{Q}$ and $\mathcal{H}$ may be simultaneously diagonalized since they commute. Similarly, if $H$ is $P$- or $Q$-symmetric with $\epsilon_q = -1$, then $\mathcal{H}$ commutes with the product $\Gamma_5\mathcal{P}$ or with $\Gamma_5\mathcal{Q}$.

Type $C$ and $K$ symmetries both act as particle–hole symmetries relating $\mathcal{H}$ to its transposed $\mathcal{H}^T$. The classification of the Hamiltonians $\mathcal{H}$ is then very simple as it follows from that of Hermitian Dirac operators. For the operators $\mathcal{H}$ to be minimal, only a type $C$ or a type $K$ symmetry can be imposed. Gauge equivalences (2.2), (3.1) leave only $\sigma = 1$ or $\sigma = \mathrm{i}\sigma_y \otimes 1$ and $\kappa = 1$ or $\kappa = \mathrm{i}\sigma_y \otimes 1$ as possible choices. We then have the correspondence

$$\begin{aligned} &\sigma = 1 && \Rightarrow \quad \mathcal{H} \in \text{class } \mathbf{7} \\ &\sigma = \mathrm{i}\sigma_y \otimes 1 && \Rightarrow \quad \mathcal{H} \in \text{class } \mathbf{8} \\ &\kappa = 1 \text{ or } \kappa = \mathrm{i}\sigma_y \otimes 1 && \Rightarrow \quad \mathcal{H} \in \text{class } \mathbf{9}_\pm. \end{aligned} \tag{3.6}$$

Though we have translated the classification of non-Hermitian $H$ into the doubled Hermitian $\mathcal{H}$, the spectra of $\mathcal{H}$ and $H$ may differ significantly. To illustrate these potential differences, consider the transformation $H \to \tilde H = -\mathrm{i}u_7 H u_7$ where $u_7 = \mathrm{diag}(1, \mathrm{i}) \otimes 1$ is the matrix we introduced in section 2. This transformation preserved the Dirac structure of the Hamiltonians but not their reality conditions. It leaves invariant $A_z$ and $A_{\bar z}$ but not the potentials since $\tilde V_\pm = -\mathrm{i}V_\mp$. Hence $H$ and $\tilde H$ should not belong to the same non-Hermitian class—and they do not have the same spectra. On the contrary, for the doubled Hamiltonians this transformation is lifted to $\mathcal{H} \to \tilde{\mathcal{H}} = \mathcal{U}\mathcal{H}\mathcal{U}^\dagger$ with $\mathcal{U} = \mathrm{diag}(-\mathrm{i}u_7, u_7^\dagger)$, so that $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are unitarily equivalent. They have identical spectra and belong to the same class.

We thus present a classification of the Dirac operators $H$ and not simply of the doubled one. As for Hermitian Dirac operators, it is the group generated by compatible discrete symmetries which is meaningful. There could be different but equivalent presentations of the same group as not all of these symmetries are independent. Indeed, the product of a type $P$ symmetry with a $C$, $K$ or $Q$ symmetry is again a $C$, $K$ or $Q$ symmetry. (See, for example, equation (2.1)). Also, the symmetries of type $Q$, $C$ or $K$ are linked as the product of any two gives a symmetry of the third type.

Let us first impose only one type of symmetry. As in section 2, up to gauge equivalence (2.2), (3.1), the solutions are

$$
\begin{array}{llll}
(\gamma = 1) & (\sigma = 1)_{\epsilon_c} & (\xi = 1)_{\epsilon_q} & (\kappa = 1)_{\epsilon_k} \\
(\gamma = \sigma_z \otimes 1) & (\sigma = \mathrm{i}\sigma_y \otimes 1)_{\epsilon_c} & (\xi = \sigma_z \otimes 1)_{\epsilon_q} & (\kappa = \mathrm{i}\sigma_y \otimes 1)_{\epsilon_k}.
\end{array}
$$

Here, each column refers to one of the possible types of symmetry. Here and below, we indicate as indices the values of $\epsilon_c$, $\epsilon_q$ or $\epsilon_k$ which matter. Thus the above list corresponds to 14 distinct classes.

Let us now impose two kinds of symmetry. First, we may require simultaneously type $P$ and $C$ symmetries. This leads to the list of six classes, from class **5** to class **9** of section 2 without any reality conditions.

Next we consider a $P$ and a $K$ symmetry. The commutativity condition for type $P$ and $K$ symmetries reads $\kappa = \pm\gamma^{-1}\kappa\gamma^*$. This is solved the same way as $\sigma = \pm\gamma^{-1}\sigma\gamma^T$ in the previous section. Thus the list of compatible type $P$ and $K$ symmetries is parallel to the list of compatible type $P$ and $C$ symmetries, only $\sigma$ is replaced by $\kappa$. Their explicit realizations are given in equations (2.21)–(2.25) but with $v_\pm^T$ replaced by $w_\pm^*$ and $w_\pm^T$ replaced by $v_\pm^*$. This corresponds to six classes.

Similarly, compatibility between type $P$ and $Q$ symmetries requires $\gamma^\dagger = \pm\xi^{-1}\gamma\xi$. Solving this constraint leads to the following compatible type $P$ and $Q$ symmetries:

$$
\begin{array}{lll}
(\gamma = 1, \xi = 1) & (\gamma = 1 \otimes 1, \xi = \sigma_z \otimes 1) & (\gamma = \sigma_z \otimes 1, \xi = \sigma_x \otimes 1)_{\epsilon_q}
\end{array}
$$
$$
(\gamma = \sigma_z \otimes 1, \xi = 1 \otimes 1)_{\epsilon_q} \cong (\gamma = \sigma_z \otimes 1, \xi = \sigma_z \otimes 1)_{-\epsilon_q}.
$$

In the second line, we have mentioned an equivalence between two solutions of the commutativity constraint. Indeed, $\xi$ of the second solution in this line is the product of $\gamma$ and $\xi$ of the first solution, so the groups generated by these solutions are identical.

We may also impose together a type $Q$ with a type $C$ symmetry. Since their product is a symmetry of type $K$ with $\epsilon_k = \epsilon_q\epsilon_c$, we are actually imposing simultaneously three compatible symmetries of different types. Any two of them generate the third. The condition for type $Q$ and $C$ symmetries to commute is $\xi^T = \pm\sigma^\dagger\xi^{-1}\sigma$; for type $C$ and $K$ symmetries this condition reads $\kappa^T\sigma^{-1}\kappa\sigma^* = \pm 1$. Up to gauge equivalences, the set of compatible type $Q$ and $C$ symmetries is then

$$
\begin{array}{ll}
(\xi = 1, \sigma = 1)_{\epsilon_q,\epsilon_c} & (\xi = \sigma_z \otimes 1, \sigma = 1 \otimes 1)_{\epsilon_q,\epsilon_c} \\
(\xi = \sigma_z \otimes 1, \sigma = \mathrm{i}\sigma_y \otimes 1)_{\epsilon_q,\epsilon_c} & (\xi = 1 \otimes 1, \sigma = \mathrm{i}\sigma_y \otimes 1)_{\epsilon_q,\epsilon_c} \\
(\xi = \sigma_z \otimes 1, \sigma = 1 \otimes \mathrm{i}\sigma_y)_{\epsilon_q,\epsilon_c} & (\xi = \sigma_z \otimes 1, \sigma = \sigma_x \otimes 1)_{\epsilon_q,\epsilon_c}.
\end{array}
$$

Finally, we may impose simultaneously a type $P$ symmetry together with two among the three types $Q$, $C$ and $K$ of symmetries. As before it is sufficient to consider only a $P$, $C$ and $Q$ symmetry. The solutions of the commutativity requirements are then

$$
\begin{array}{l}
(\gamma = 1 \otimes 1, \xi = 1 \otimes 1, \sigma = 1 \otimes 1 \text{ or } \mathrm{i}\sigma_y \otimes 1) \\
(\gamma = 1 \otimes 1, \xi = \sigma_z \otimes 1, \sigma = 1 \otimes 1, \sigma = 1 \otimes \mathrm{i}\sigma_y, \mathrm{i}\sigma_y \otimes 1 \text{ or } \sigma_x \otimes 1) \\
(\gamma = \sigma_z \otimes 1, \xi = 1 \otimes 1, \sigma = 1 \otimes 1 \text{ or } 1 \otimes \mathrm{i}\sigma_y)_{\epsilon_q}
\end{array}
$$

$$(\gamma = \sigma_z \otimes 1, \xi = 1 \otimes 1, \sigma = \sigma_x \otimes 1)_{\epsilon_q,\epsilon_c}$$
$$(\gamma = \sigma_z \otimes 1, \xi = \sigma_x \otimes 1, \sigma = 1 \otimes 1, 1 \otimes \mathrm{i}\sigma_y, \sigma_x \otimes 1, \text{ or } \sigma_x \otimes \mathrm{i}\sigma_y)_{\epsilon_q,\epsilon_c}.$$

Here, each choice of $\sigma$ corresponds to a different class.

Considering more combinations of the four different kinds of symmetries would not lead to new minimal classes, because in such case we would always be able to construct some matrix commuting with the Hamiltonians and preserving their Dirac structure.

For each set of compatible symmetries, one has to choose the signs $\epsilon_q$, $\epsilon_c$ and $\epsilon_k$ to specify the classes. These signs are used to index the solutions in the above lists. The absence of one of these indices means that the corresponding solution is independent of that index. The grand total is 87 classes.

It is straightforward to determine the form of $H$ for each of the above symmetry classes; however, there is little motivation to list the details here. Let us just describe a simple example, corresponding to imposing a symmetry of type $K$, relating $H$ to its complex conjugate, with $\kappa = 1$ and $\epsilon_k = +1$. Then, relations (3.3) yield $A_{\bar{z}} = -A_z^*$ and $V_\pm = \pm V_\pm^*$, such that $V_+$ is real and $V_-$ imaginary. As a consequence, the Dirac Hamiltonian may be written as

$$\begin{pmatrix} M & -\mathrm{i}\partial_{\bar{z}} - A_z^* \\ -\mathrm{i}\partial_z + A_z & M^* \end{pmatrix}$$

with $M = V_+ + V_-$. This class, studied e.g. in reference [31], is closely related to the random $XY$ model. The doubled Hamiltonian belongs to class $\mathbf{9}_-$.

Having performed the above classification we can easily specialize it to random non-Hermitian matrices with no Dirac structure. As for random Hermitian matrices (see the appendix for a summary), the above classes with $\gamma = 1$ are trivial and should be thrown away. The choice of the signs $\epsilon_q$ and $\epsilon_k$ is also irrelevant since they can be absorbed into $H \to \mathrm{i}H$ which is now allowed since no Dirac structure is imposed. Altogether this gives 43 classes which will be described in greater detail in [32].

## Appendix

For completeness—and to explain table 1—we recall the definition of random Hermitian matrix ensembles [15]. We denote by small letters quantities referring to random matrices. Let $h = h^\dagger$ be a Hermitian matrix and $p$ and $c$ be the operators implementing the discrete symmetries as in equations (2.14), (2.15): $h \to -php^{-1}$, $h \to \epsilon_c c h^T c^{-1}$. For each random matrix class, the defining relations for $p$ and $c$ are summarized in table A1.

**Table A1.** Random matrix classes.

| Random matrix classes | Discrete symmetry relations |
|---|---|
| **A** | $h = h^\dagger$ |
| $\mathbf{A}_{\mathrm{I}}$ | $c^T = c, \epsilon_c = +$ |
| $\mathbf{A}_{\mathrm{II}}$ | $c^T = -c, \epsilon_c = +$ |
| $\mathbf{A}_{\mathrm{III}}$ | $p^2 = 1$ |
| **C** | $c^T = -c, \epsilon_c = -$ |
| **D** | $c^T = c, \epsilon_c = -$ |
| $\mathbf{D}_{\mathrm{I}}$ | $p^2 = 1, c^T = c, \epsilon_c = \pm, pcp^T = c$ |
| $\mathbf{C}_{\mathrm{II}}$ | $p^2 = 1, c^T = -c, \epsilon_c = \pm, pcp^T = c$ |
| $\mathbf{C}_{\mathrm{I}}$ | $p^2 = 1, c^T = \pm c, \epsilon_c = \pm, pcp^T = -c$ |
| $\mathbf{D}_{\mathrm{III}}$ | $p^2 = 1, c^T = \pm c, \epsilon_c = \mp, pcp^T = -c$ |

In each of the last four lines of table A1, one may equivalently choose either the upper or the lower signs, since this simply corresponds to choosing two equivalent presentations of the same class.

To compare with the classification of Dirac fermions, it is useful to note that in the latter case the operators $P$ and $C$ may be written as

$$P = \tau_z \otimes \gamma \qquad C = \mathrm{i}\tau_y \otimes \sigma \quad \text{for} \quad \epsilon_c = + \qquad C = \tau_x \otimes \sigma \quad \text{for} \quad \epsilon_c = -.$$

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